Microscale Gravity Wave Physics and its Relevance to the Macro Middle Atmosphere (*Scale Wars*)

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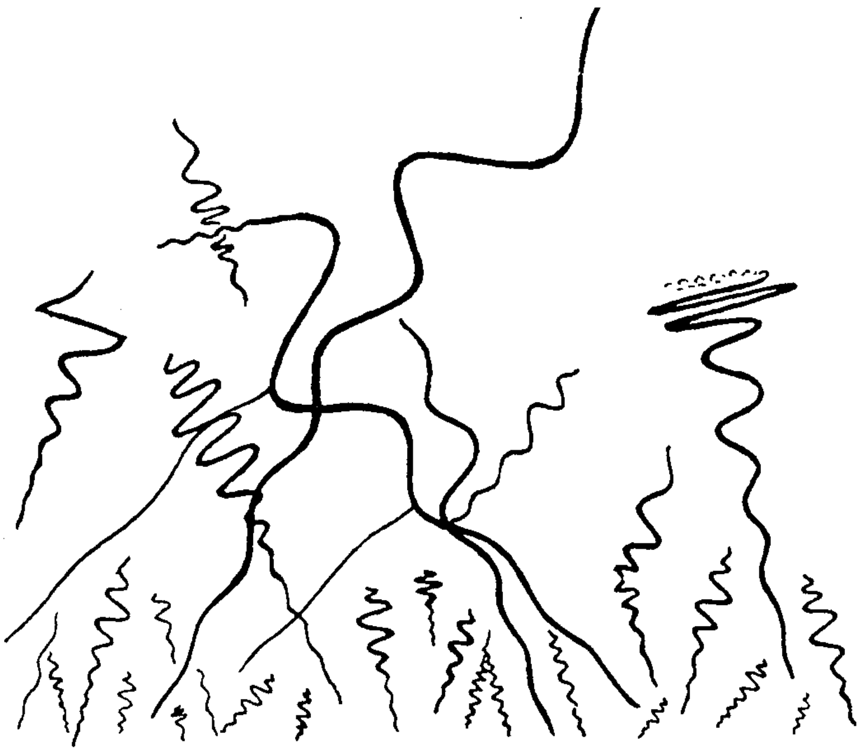
or to paraphrase Colin Hines:

A way cool toy for the



Hines' *The Atmosphere*

http://pcl.physics.uwo.ca

Outline

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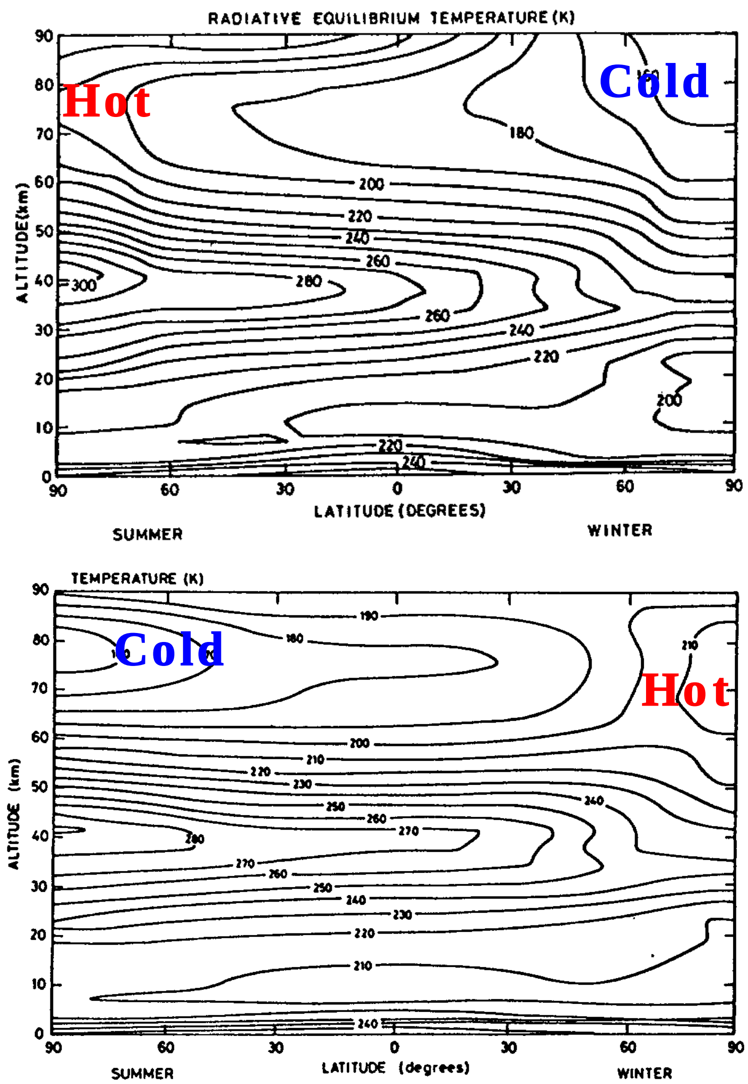
IX. Why MAM has to worry about “tiny” waves

I. Why MAM has to worry about “tiny” waves

The troposphere is approximately in radiative equilibrium. Not so the middle atmosphere, due to gravity waves. How do gravity waves cause this dramatic departure from radiative equilibrium (figure from *Fritts et al.* 1984)?

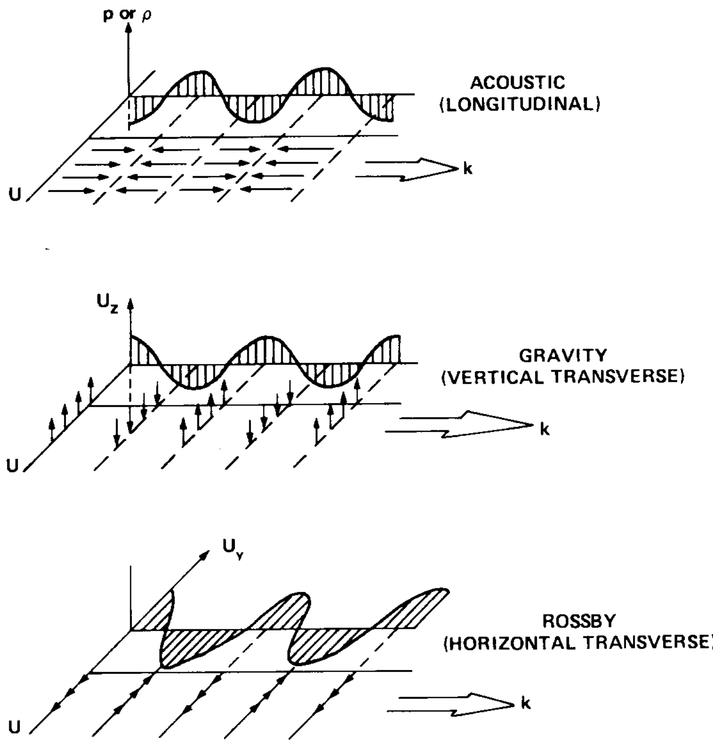
Calculated

Observed



II. The Importance of Gravity

Here is a picture showing three important type of atmospheric waves (*Beer* 1974). Gravity and Rossby waves



are transverse waves while acoustic waves are longitudinal waves.

The importance of gravity as the restoring force for gravity waves is illustrated by the *Froude number*:

$$\mathit{Froude}\ \# = \frac{\text{pressure}}{\text{gravity}} = \frac{|\bar{\boldsymbol{U}}|}{\sqrt{gL}} \tag{1}$$

where $|\bar{\boldsymbol{U}}|$ is the magnitude of the mean wind (with components u(zonal), v (meridional) and w (vertical)), g is the acceleration due to gravity and L a characteristic length scale. The appropriate scale length is the scale height, H, as defined in the *Barometric Law* for pressure

$$p = p_{ref}e^{-z/H} \tag{2}$$

$$H = \frac{k_b T}{Mg} \tag{3}$$

where p_{ref} is a reference pressure given a some height z, T is the temperature, M the mean molecular mass and k_b Boltzmann's constant.

For typical values of the wind (that is between 1 and 100 m/s) the *Froude #* varies between 0.005 and 0.5.

- *Froude #* << 1

  Hydrostatics (gravity >> pressure and inertial forces)
- *Froude #* < 1

  Gravity waves
- *Froude #* ≈ 1

  Acoustic gravity waves
- *Froude #* >> 1

  Acoustic waves (low frequencies, infrasonic waves)

Some more terminology

- External Waves (no vertical phase)

  Surface waves (at interface, e.g. waves on the surface of a lake)

  Evanescent waves (exist independent of boundary and can propagate within body of a fluid)

- Internal Waves (have vertical phase)

  Gravity waves

  Acoustic waves

  Rossby waves

III. Stability

The *environmental lapse rate*, determined from an atmosphere's temperature profile, is given by

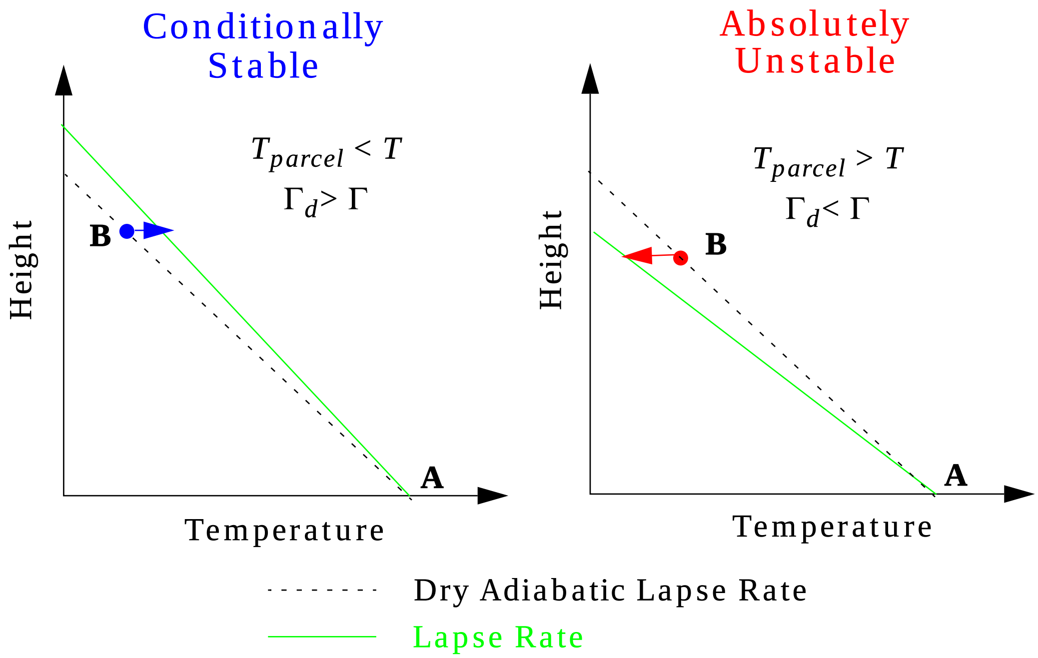
$$\Gamma = -\frac{dT}{dz}. \qquad (4)$$

In an atmosphere with no condensable gases the *dry adiabatic lapse rate*, Γ_d, is the largest temperature change the atmosphere can sustain and remain stable to convection. In general

$$\Gamma_d = \frac{g}{c_p} \qquad (5)$$

where c_p is the specific heat of air at constant pressure. For Earth $\Gamma_d = 9.8$ K/km.

The following figure illustrates the physics of the stability process. An air parcel is lifted from point **A** adiabatically. Its



temperature must decrease at the dry adiabatic lapse rate (the dashed line). At some height, **B**, let's compare its temperature, T_{parcel}, to the air temperature, T. The air parcel's temperature is less than the local temperature; thus, it is denser than the surrounding air.

? What must it do?

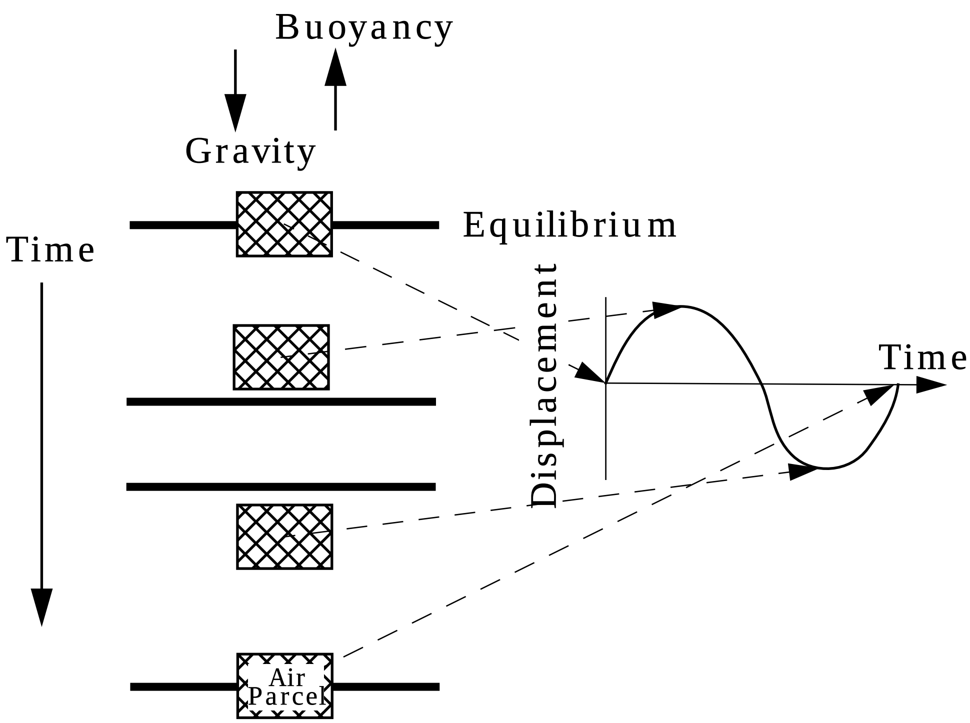
Subside, since an air parcel cooler than its surroundings must descend. The atmosphere is then stable against convection. If convection tries to lift an air parcel it falls right back to its original position. In terms of lapse rates, the lapse rate of the atmosphere (Γ) is less than the dry adiabatic lapse rate. The condition $\Gamma < \Gamma_d$, is called *conditionally stable*.

On the right side of the figure the same air parcel is lifted from **A** to **B** in an atmosphere where the lapse rate is greater than the dry adiabatic lapse rate. At **B** the temperature of the air parcel is now greater than the local air temperature.

? What must this air parcel do?

Rise, because the hot air is less dense than the surrounding air. This atmosphere is *absolutely unstable*, and convection will occur. In terms of lapse rates, the lapse rate of the atmosphere is greater than the dry adiabatic lapse rate ($\Gamma > \Gamma_d$). A lapse rate that is greater than the dry adiabatic lapse rate is also called a *superadiabatic lapse rate*. If the lapse rate were exactly equal to the dry adiabatic lapse rate the atmosphere would be in equilibrium. If the local temperature increased the air would begin to become unstable.

For stable air, the frequency of this oscillation is called the *Brunt Väisälä frequency*, though the terms BV frequency and buoyancy frequency are common. Consider the following picture. The displaced parcel will feel a restoring



force due to gravity. From *hydrostatic equilibrium*,

$$\frac{dp}{dz} = -\rho g \qquad (6)$$

and since the change in pressure for the air parcel and its surroundings is the same, the force per mass is given by

$$g\left(\frac{\rho - \rho_0}{\rho}\right) = g\left(\frac{T_0 - T}{T}\right) = \frac{g}{T}(\Gamma_d - \Gamma)\delta z \qquad (7)$$

where we have used the Ideal Gas Law and the definition of lapse rate for a parcel displacement δz. From Newton's Second Law

$$\frac{d^2}{dt^2}(\delta z) = \frac{g}{T}(\Gamma_d - \Gamma)\delta z \qquad (8)$$

we obtain the equation for a simple harmonic oscillator whose angular frequency is the Brunt Väisälä frequency, N,

$$N^2 = \frac{g}{T}(\Gamma_d - \Gamma). \qquad (9)$$

Stability conditions:

- $N^2 > 0$; stable
- $N^2 < 0$; unstable

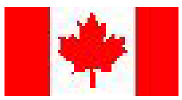
In the middle atmosphere the BV period is $\tau_{BV} = \frac{2\pi}{N} \approx 5.5 \text{ min}$ compared to the acoustic angular frequency

$$\omega_a = \frac{C}{2H} \to \tau_a \approx 4.75 \text{ min}. \qquad (10)$$

Hence, a frequency "gap" exists between the two types of internal waves, infrasonic and gravity waves.

IV. Dispersion and Polarization Equations

A dispersion equation describes how the frequency of a wave depends on its wavenumber. The simplest situation for gravity waves is the dispersion relation derived from the linearized equation of motion, adiabatic equation of state and conservation of mass equation.



Let's follow the lead of Colin Hines in his seminal paper on gravity waves.[1]

Equation of Motion: $$\rho_0\left(\frac{\partial \boldsymbol{U}}{\partial t}\right) = \rho \boldsymbol{g} - \nabla p \qquad (11)$$

State Equation: $$\frac{\partial p}{\partial t} + \boldsymbol{U} \cdot \nabla p_0 = C^2\left[\frac{\partial \rho}{\partial t} + \boldsymbol{U} \cdot \nabla \rho_0\right] \qquad (12)$$

Conservation of Mass: $$\frac{\partial \rho}{\partial t} + \boldsymbol{U} \cdot \nabla \rho_0 + \rho_0 \cdot \nabla \bullet \boldsymbol{U} = 0 \qquad (13)$$

Here ρ is density and $C^2 = \gamma \frac{p_0}{\rho_0}$ is square of the sound speed.

The '0' subscripts refer to background values of the parameter, e.g.

1. Paper 7 in Hines' *Upper Atmosphere in Motion*, which you will go buy on Monday. Hines' original paper was published in the *Canadian Journal of Physics* (1960). This isn't Dr. Hines only seminal paper; during the same period he and Ian Axford developed a model for convection in a closed magnetosphere!

$$p' = p - p_0 \qquad (14)$$

where the primed quantities are small perturbations.

These equations can be solved by Fourier methods with solutions

$$\left(\frac{p'}{p}\right)\frac{1}{P} = \left(\frac{\rho'}{\rho}\right)\frac{1}{R} = \frac{u'}{X} = \frac{w'}{Z} = Ae^{i(\omega t - kx - mz)} \qquad (15)$$

Here P, R, X, Z and A are complex coefficients and we have assumed an isothermal atmosphere with no background wind (i.e. $\boldsymbol{U_0} = 0$). The angular frequency is ω, while the angular wavenumbers in the horizontal is k and in the vertical m (i.e. $m = 2\pi/\lambda_z$ where λ_z is the vertical wavelength).

The simultaneous solution of (11)-(13) gives a matrix whose determinant is the *dispersion equation*:

$$(\omega^2 - \omega_a^2)\frac{\omega}{C^2} - \omega^2(k^2 + m^2) + N^2k^2 = 0 \qquad (16)$$

We can calculate the "asymptotic" conditions assuming high vertical wavenumber

$$m^2 » \frac{1}{4H^2} \rightarrow \lambda_z < 14 \text{ km} \tag{17}$$

and low frequencies

$$\omega « \frac{g}{C} \rightarrow \tau » 4 \text{ min}. \tag{18}$$

Applying the first condition simplifies the dispersion relation to

$$\omega^2 m^2 \cong (N^2 - \omega^2)k^2, \tag{19}$$

while the second condition further simplifies it to

$$\omega^2 m^2 \cong N^2 k^2. \tag{20}$$

We can now solve for the constants *P*, *R*, *X* and *Z* in (15), which are called the *polarization equations*

$$P = \gamma\omega^2 m \tag{21}$$

$$R = i(\gamma - 1)gk^2 \tag{22}$$

$$X = \omega k m C^2 \tag{23}$$

$$Z = -\omega k^2 C^2. \tag{24}$$

The polarization equations describe the phase relationship between pressure, density and wind. For instance,

$$\frac{Z}{X} = \frac{w}{u} = -\frac{k}{m} \tag{25}$$

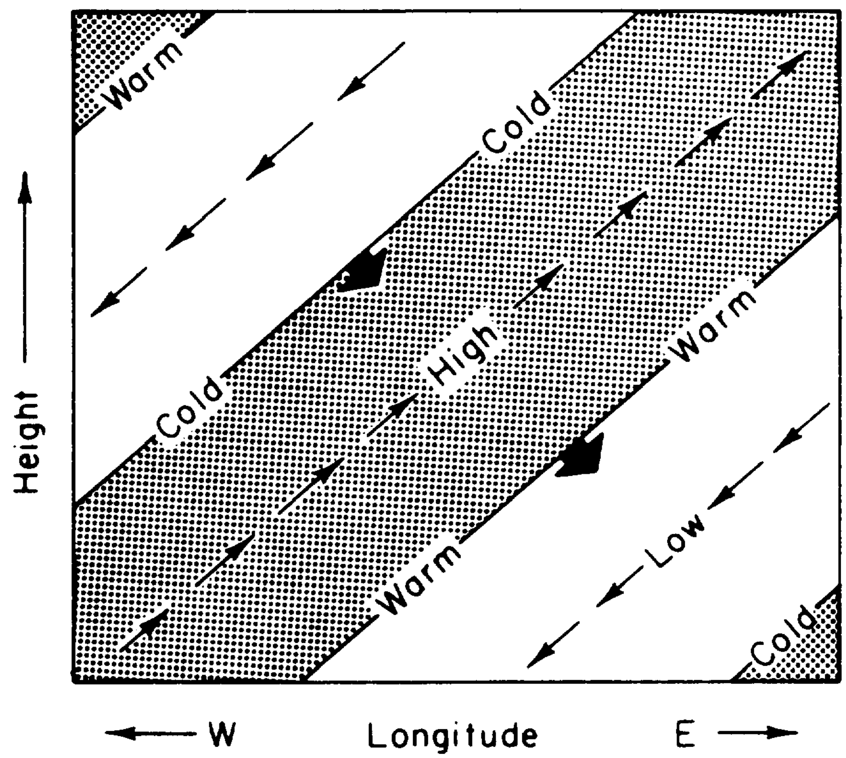
and

$$\pm\frac{R}{X} = \frac{\rho'}{\rho_0 u} = i\frac{N}{g}. \tag{26}$$

Equation (25) could be used for estimating the vertical wind perturbation caused by a gravity wave. A Doppler lidar or a Rayleigh-scatter lidar and a MST radar would measure *u*, *k* and *m* and use (25) to infer *w*.

Equation (26) tells us that the density and horizontal wind perturbations are 90^{o} out of phase. This relation is also useful for converting wind perturbation measurements (as made by a MST radar) to density perturbations and vice versa (for density perturbation measurements made by, say, a Rayleigh-scatter lidar).

These relationships are show in the following figure from *Holton's* book. Here High and Low refer to pressure, warm



and cold to temperature, the thin arrows show the perturbed wind field, the short thick arrows indicate the wave's phase velocity and the shaded regions show upward motion.

Note:

- You can show from the state equation that to zeroth order density fluctuations are equal in magnitude and opposite in sign to temperature fluctuations (physically why?!).

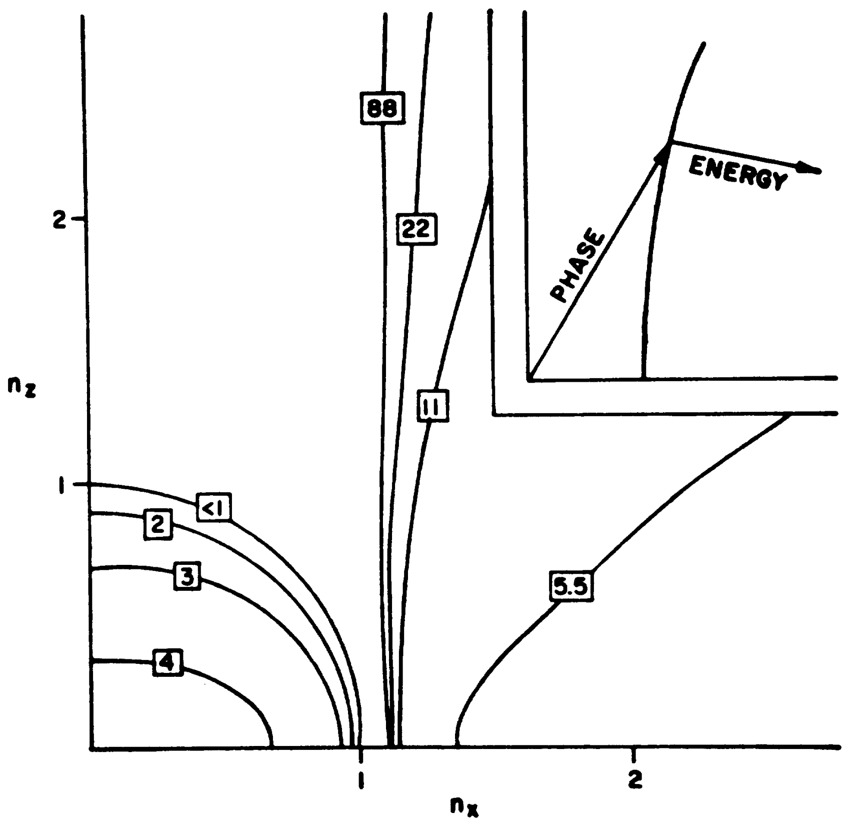
  Note that the temperature fluctuations are 90^{o} out of phase with the wind fluctuations.
- High pressure corresponds with upward motion.

V. Phase and Group Velocity

The following figure from Hines' seminal paper shows contours of constant wave period in a coordinate space of phase velocity relative to the sound speed,

$$n_x = \left(\frac{k}{\omega}\right)C = \frac{C}{v_{px}};\; n_z = \left(\frac{m}{\omega}\right)C = \frac{C}{v_{pz}}. \tag{27}$$

similar to the optical index of refraction. Note:



- for acoustic waves the contours are nearly circular and v_p is parallel to the energy dissipation
- for gravity waves the contours become constant in v_{px} and v_p is perpendicular to the energy dissipation.

Energy propagates at the *group velocity*

$$v_g = \frac{d\omega}{d\kappa} \tag{28}$$

where $\kappa(k,l,m)$ is the wave vector.

From the dispersion relation please show for yourself that

$$\boldsymbol{v_p} = \pm\frac{Nk}{(k^2+m^2)}\left(\frac{k}{m}\hat{x}+\hat{z}\right) \tag{29}$$

$$\boldsymbol{v_g} = \pm\frac{N}{m}\left(\hat{x}\mp\frac{k}{m}\hat{z}\right) \tag{30}$$

and most importantly

$$\boldsymbol{v_p}\bullet\boldsymbol{v_g} = 0, \tag{31}$$

as expected for a transverse wave. Also notice that the phase velocity is downward for upward propagating energy and vice versa.

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MAM Summer School
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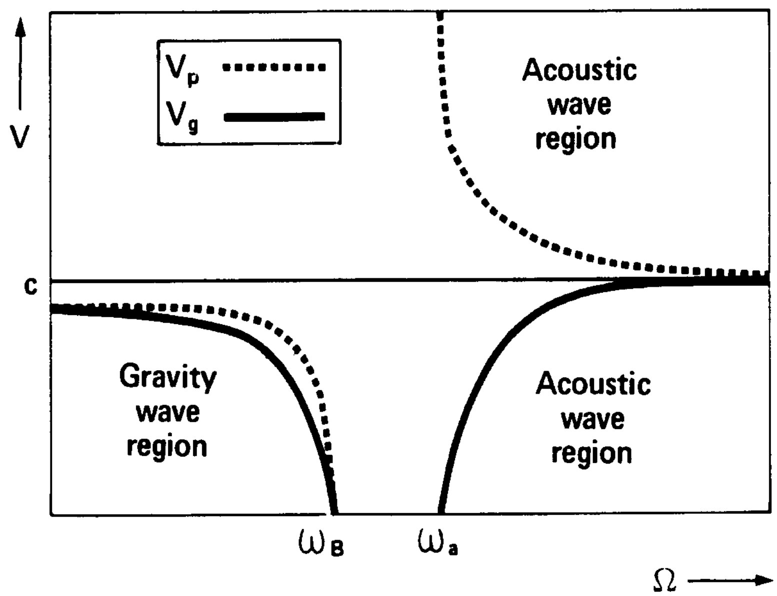
The allowed regions of phase and group velocities for gravity and acoustic waves is shown below (*Beer* 1974). Note:



- the phase and group velocity of gravity waves is less than the sound speed, with the phase velocity slightly larger than the group velocity at high frequencies
- acoustic waves have phase velocities above the sound speed, and the vertical group velocity can never exceed the vertical phase velocity.

VI. Effect of the Background Wind

If we had included a background wind in the calculation we would have obtained a dispersion relation

$$\hat{\omega}^2(k^2+m^2) - N^2k^2 = 0 \qquad (32)$$

where

$$\hat{\omega} = \omega - \bar{U}k = \pm\frac{Nk}{\sqrt{k^2+m^2}}. \qquad (33)$$

Hence, the frequency we observe from the ground (or space), the *observed frequency*, is not the frequency of interest, $\hat{\omega}$, the *intrinsic frequency* of the wave. Doppler shifting can have an important effect on the interpretation of gravity wave measurements, and will be the final judge on whether a wave survives or is surrenders its momentum flux to the mean flow.

VII. Conservation of Energy

For the linear theory under discussion it is assumed that there is an equipartition between kinetic and potential energy. This assumption appears reasonable at lower heights, but begins to break down in the upper mesosphere and lower thermosphere.

The potential energy density of the wave is

$$PE = \frac{1}{2} g \left(\frac{\bar{\rho}'^2}{\rho_0} \right) \frac{1}{N^2} \tag{34}$$

where $\bar{\rho}'^2$ is the square of the mean density fluctuation, while the kinetic energy density is

$$KE = \frac{1}{2} \rho_0 (u'^2 + v'^2 + w'^2) = \frac{1}{2} \rho_0 U'^2 . \tag{35}$$

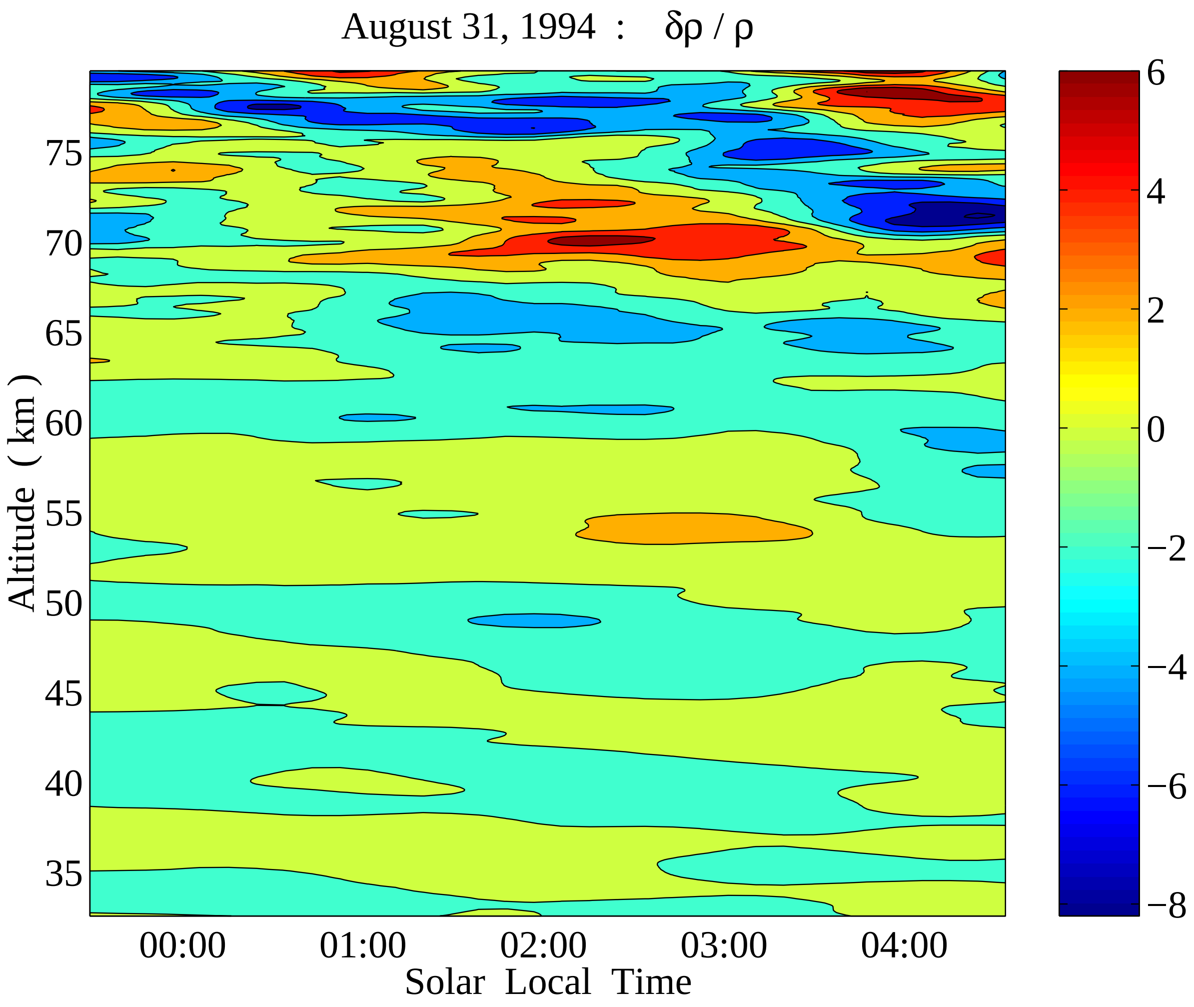
Conservation of energy tells us how gravity waves grow with height. From the Barometric Law we can substitute for ρ_0 and set the *KE* equal to a constant. Do this and you will find that

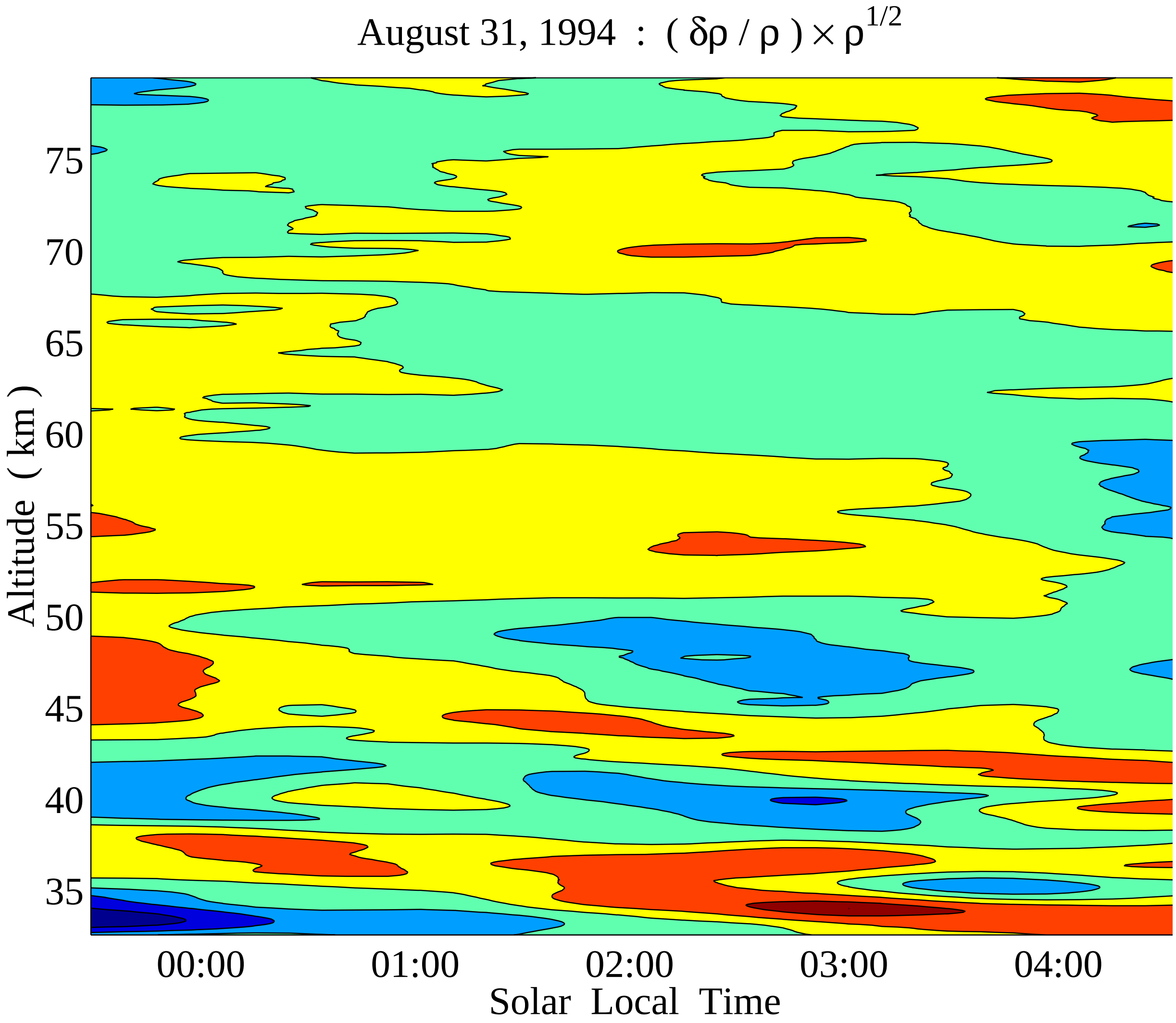
$$U'^2 \propto e^{z/2H} \propto \frac{1}{\sqrt{\rho_0}} . \tag{36}$$

A gravity wave grows a factor of *e* in 2 scale heights (10 times in 32 km). You would obtain a similar result using the expression for potential energy.

Though the initial pressure disturbance that launches a wave may be small, the disturbance can grow 1000 times if it reaches the upper mesosphere!

Here is an example of density perturbation measurements from the Purple Crow Lidar with and without scaling the perturbations by the square root of density.[1]



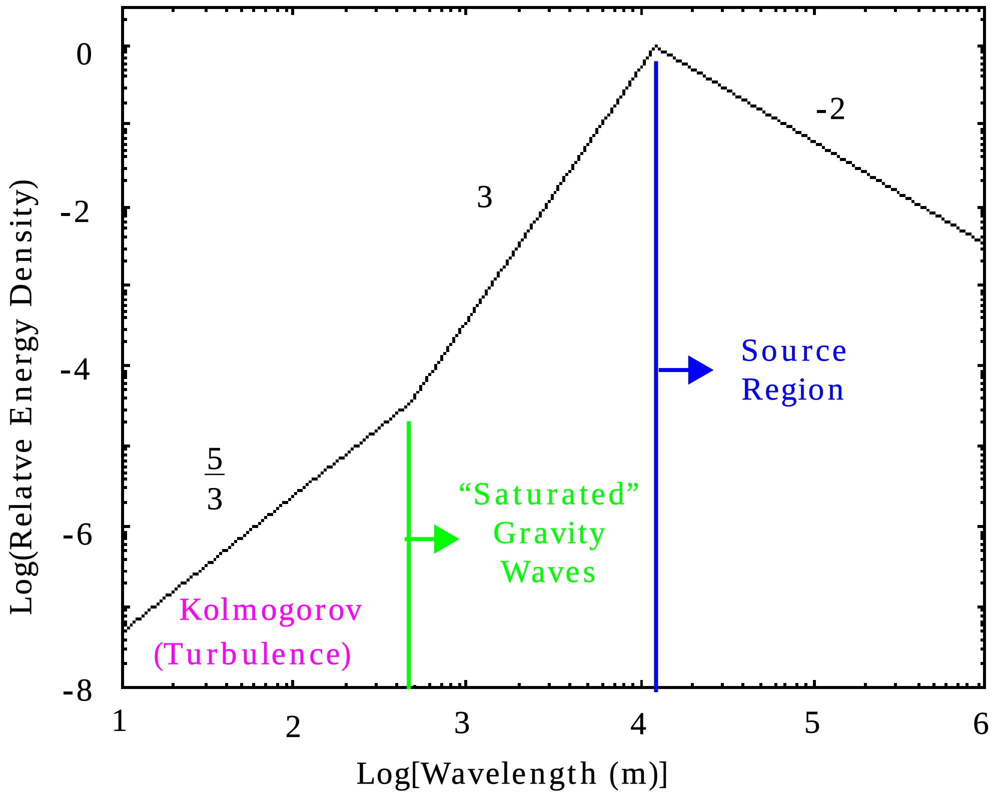


The large-scale wave structures are much clearer when the perturbations are scaled by the square root of the density.

1. FYI: for these contours Δz = 48 m, Δt = 9 min. The measurements have been filtered in height with a 1008 m cutoff and in time with a 63 min cutoff.

VIII. Vertical Wavenumber Spectrum

The vertical wavenumber spectrum (sometimes called the "m spectrum") of gravity waves has three distinct regions.



- Source Region

  Sources for gravity waves include:

1. Weather fronts (convection).
2. Storms and severe weather.
3. The polar front jet stream.
4. Topography.
5. Breaking tidal and planetary waves.
6. Explosions.

- "Saturated" or Tail Region
- Kolmogorov Region

  It is generally thought the slope of this region is due to the cascade of gravity waves into turbulent eddies

Linear Saturation Theory is

- easy to grasp
- easy to parameterize to improve the calculations of GCMs
- *wrong!*

Gravity waves are unstable to convection and *break* (that is *saturate* in amplitude) when $\Gamma > \Gamma_d$, or equivalently when

$$u > \left|v_{px} - \overline{U}\right| . \tag{7}$$

From the dispersion relation (assuming $m^2 > k^2$)

$$|m| = \frac{N}{\left|v_{px} - \overline{U}\right|}, \tag{8}$$

so

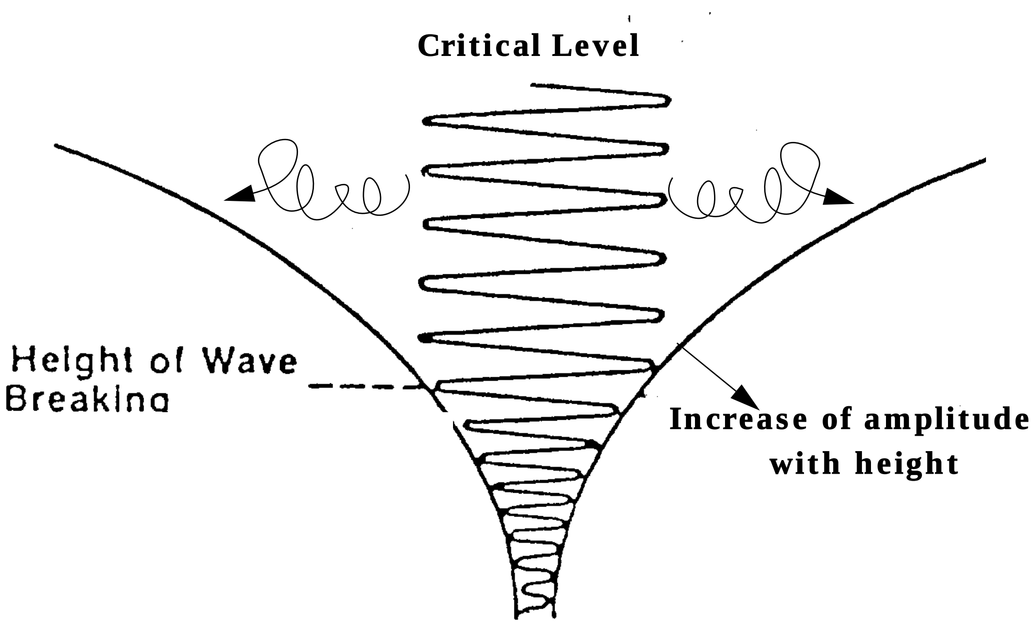
$$u > \left|v_{px} - \overline{U}\right| = \frac{N}{|m|} \tag{9}$$

is the saturation limit for discrete waves. For instance, if $N = (2\pi/5\ \text{min})$ and $m = (2\pi/10\ \text{km})$, then the saturation amplitude for wave breaking is 30 m/s.

After the wave breaks it amuses itself by throwing off its excess energy as turbulent eddies which cause mixing.

When the wave's phase speed equals the background wind speed the wave is finally able to deposit its momentum to the mean flow. We call the height at which $|v_{px} - \overline{U}| = 0$ a *critical level.*

Here is a cartoon of the process, adapted from *Fritts et al.*



For an ensemble of sinusoids the total power (also termed the variance) is

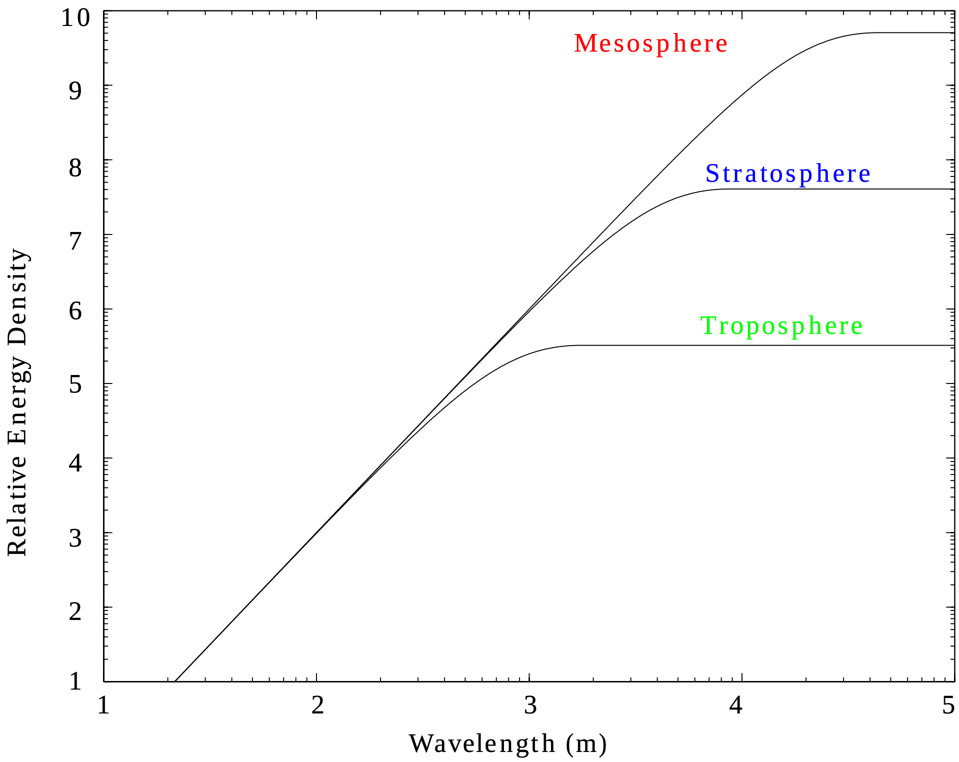
$$\frac{N^2}{2m} \tag{10}$$

Scaling arguments suggest a spectral form proportional to m^{-1}, so the power spectral density of the vertical wavenumber spectrum is

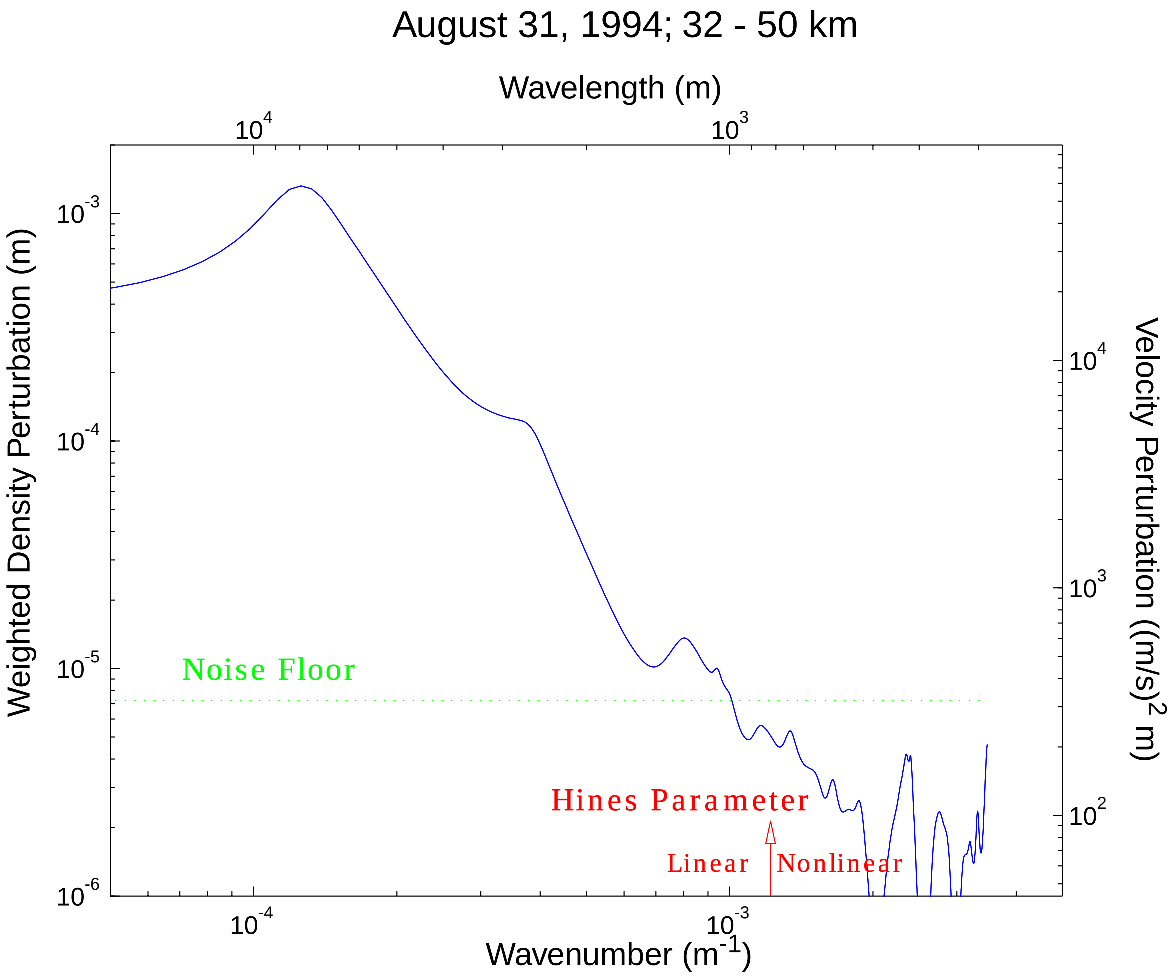
$$F(m) \propto \frac{N^2}{2m^3} \qquad (11)$$

- Tail spectrum has a slope of 3

The tail spectrum increases in variance with height, and the beginning of the tail spectrum moves to smaller wavenumber with height (from *Smith et al.*)



WARNING: Real spectrum averaged over short periods (less than a few hours) have many complicated features not addressed by this theory.



The above spectrum is averaged over the approximately 5 hour period shown in the previous plots of $(\Delta\rho/\rho)\rho^{1/2}$.

"Modern" theories include Doppler spread (*Hines*), diffusive filtering (*Gardner*) and scale-dependent diffusion (*Weinstock*).

IX. Why MAM has to worry about "tiny" waves

Recall that in the mesosphere and lower thermosphere large temperature differences exist between the observed and radiative equilibrium temperatures. At 80 km:

Season	T_{rad}(K)	T_{obs}(K)	T_{obs}-T_{rad} (K)
Summer	210	160	-60
Winter	130	220	90

Winter: $T_{rad} < T_{obs}$

Summer: $T_{rad} > T_{obs}$

The temperature difference causes the air to rise in the summer pole and subside in the winter pole.

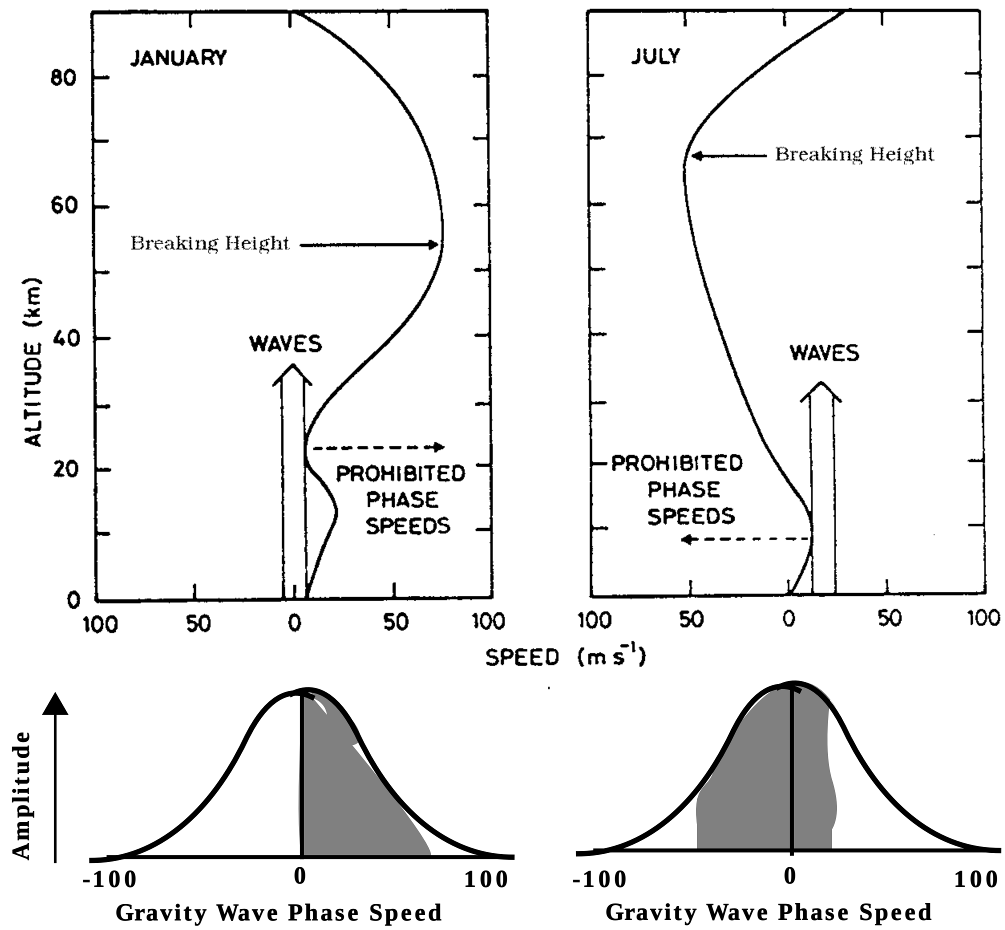
The transformed Eulerian mean equation for the vertical residual circulation, $\bar{w}^*$, for steady-state conditions is:

$$N^2\bar{w}^* = -r(T_{obs} - T_{rad}) \qquad (12)$$

where r is Newtonian cooling coefficient.

- The equation agrees with out intuition; cold air moves $_{down}$ in winter and hot air moves ^{up} in summer

Consider the following figure from adapted from *Brasseur and Solomon*.



In winter:

- wave breaking height is lower

WHY? Mean wind removes most eastward waves. Large amplitude westward waves cannot grow much without becoming unstable and breaking.

- mostly westward momentum is deposited (wave drag < 0)

In summer

- wave breaking height is higher than winter

WHY? Mean wind removes most of westward waves and most of large amplitude eastward waves. Smaller amplitude eastward waves can grow more before breaking and thus get to higher altitudes.

- eastward momentum is deposited (wave drag > 0), but not as much as in winter since mean wind does not filter out all of the westward travelling waves.

The wave drag, *F*, creates a meridional wind (actually the meridional residual circulation) since at steady state

$$-f\bar{v}^* = F \qquad (13)$$

where *f* is the Coriolis parameter.

Winter: $F < 0$, residual meridional wind is poleward.

Summer: $F > 0$ so residual wind is equatorward.

What's the linkage between the vertical and meridional wind (and hence, wave drag!)?

The downward control principle tells us that larger wave drag leads to a more air moving upward or downward.

Wave drag is larger in winter due to momentum contributions from the westward travelling waves.

This result is constant with the fact that the difference between T_{rad} and T_{obs} is larger in winter than summer.

So, more air is transported downward in winter than upward in summer.

Weaker wave drag in summer is due to momentum contributions from both eastward and westward travelling waves causing drag in opposite directions, causing a smaller diabatic circulation.

- The deposition of energy from the lower atmosphere by gravity waves explains the mystery of why the summer polar mesopause is cooler than the winter polar mesopause.

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